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Revolution Of the Electric Field with Darboux Frame

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In this study, we examined the geometric phase related to Darboux frame and analyzed its relation to the motion of the polarized light wave and electromagnetic trajectories in an optical fiber in Minkowski 3-space. The study is presented as follows: The first section is the part that contains the only developments on the subject. In the second section, the theoretical information used in the publication is given. The third section investigates the geometric phase of the polarization plane of a light wave traveling in an optical fiber through Darboux frame in Minkowski 3-space. The fourth section shows some examples using the Maple program.

Keywords: *Magnetic field, Fermi-walker derivative, Magnetic theory*

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1. Introduction

Mathematics has recently gained importance in studies related to optics. Geometric approaches and discussions about Rytov's law, which have a great place in optics and electromagnetic theory, were first included in ([1],[2],[3]) publications. With the Berry's phase, which is also known as the geometric phase and emerged after the study of the topological phase, studies on geometric approaches have begun to increase. For instance, it was first discussed by Ross et al. that the motion of a particle could be studied using the whole theory of geometric approximations. (See [5],[6]). With this development, subjects such as curves and curvature, motion, and trajectories, which are important in terms of geometry, have been discussed by many authors for electromagnetic theory. (See [7],[8],[9],[10],[11]). In ([12],[13]), in the light of these studies, the authors examined the movement of polarized light and its orbit in the riemman space, which is the fundamental space. however, in ([15]), the theory of homotetic motion was studied together with quaternions to study magnetic and electromagnetic trajectories. Thanks to all these studies, it was possible to examine this geometric phase in other geometric structures. for example, it has been studied in hyperbolic plane in ([19]). Adachi furthered these studies and investigated their counterparts in complex projective space, ([17],[18]). Studies including geometrical approaches of magnetic and electromagnetic theory have been carried out in other important space as well. ([21]). Along with all the studies done until that time, the most fundamental publications in which magnetic and electromagnetic trajectories were investigated are ([22],[23],[16]). Then, in ([20],[24],[25]), the authors gave characterizations of the magnetic curves in 3-dimensional Euclidean 3-space E^3 and Lorentzian 3-space E_1^3 .

2. Fundamental backgrounds

Let γ be non-null curve lying on the non-null surface M and $\{T, Q, N\}$ be Darboux frame on the surface M in Minkowski 3-space R_3^1 with standard metric of R_3^1 ;

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of R_3^1.

T is unit tangent vector of the curve; N is the unit normal vector of surface M and Q is a unit vector given by $Q = T \wedge_L N$.

The derivative of the Darboux frame is given by;

$$\begin{pmatrix} T' \\ Q' \\ N' \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_2 k_g & \varepsilon_3 k_n \\ -\varepsilon_1 k_g & 0 & \varepsilon_3 \tau_g \\ -\varepsilon_1 k_n & -\varepsilon_2 \tau_g & 0 \end{pmatrix} \begin{pmatrix} T \\ Q \\ N \end{pmatrix} \quad (1)$$

where;

$$\langle T, T\rangle_L = \varepsilon_1 \quad \langle Q, Q\rangle_L = \varepsilon_2 \quad \langle N, N\rangle_L = \varepsilon_3.$$

Also;

$$T \wedge_L Q = \varepsilon_3 N \quad Q \wedge_L N = \varepsilon_1 T \quad N \wedge_L T = \varepsilon_2 Q .$$

Respectively k_g, k_n and τ_g are the geodesic curvature, normal curvature, and geodesic torsion in Minkowski 3-space.
The magnetic field is a vector field and mathematically corresponds to the vector field div=0 in 3D Riemannian manifolds. The force acting on the magnetic field is called the Lorentz force is defined by the skew symmetric operator ϕ and is given as follows

$$\Phi(X) = V \times X, \quad (2)$$

The trajectory formed as a result of the particle moving with this force acting on the particle under the influence of the magnetic field is called the magnetic trajectory. The magnetic curves of the magnetic vector field V provide the following equation

$$\Phi(t) = V \times t = \nabla_t t, \quad (3)$$

([22],[23]).

3. A Geometric Phase Model of The Polarized Light Wave in The Optical Fiber Through Darboux Frame

An optical fiber can be defined by a non-null curve in Semi-Riemannian manifold. Considering β is a space curve. The direction of the state of the polarized light is defined via the derivative of E. Thus, along with the optical fiber the direction of E can be written as the linear combination of the Darboux frame fields in Minkowski 3-space. Then we can write the following

$$\frac{dE}{ds} = \lambda_1 T(s) + \lambda_2 Y(s) + \lambda_3 Z(s), \quad (4)$$

where λ_i, i=1,2,3 are differentiable functions.
Next, the direction of the state of the polarized light was examined the angle made by the electric field with the fields of the frame in three different cases according to the right angle.

3.1. A geometric phase model of the polarized light wave in the optical fiber through Darboux frame $E \perp T$

Case 1: Let we suppose that E make a right angle with T. Thus, we have;

$$\langle E, T\rangle = 0. \quad (5)$$

If we take the derivative of (5) and take into account the (4) and (1) equations, make the necessary calculations, and assume there is no mechanism lossin the optic fiber because of absorption, we have $\langle E, E\rangle = k$, where k is constant, we can write;

$$E = \varepsilon_2 \langle E, Q\rangle Q + \varepsilon_3 \langle E, N\rangle N. \quad (6)$$

When necessary calculations are made, we can get

$$\frac{dE}{dt} = (-k_g \varepsilon_1 \varepsilon_2 \langle E, Q\rangle - k_n \varepsilon_1 \varepsilon_3 \langle E, N\rangle) T + \lambda \langle E, N\rangle Q - \lambda \langle E, Q\rangle N. \quad (7)$$

The λ part of Eq (7) shows the rotation around the principal tangent vector T. If we assume that T is parallel transported (i.e., $\lambda=0$), then we find;

$$\frac{dE}{dt} = (-k_g \varepsilon_1 \varepsilon_2 \langle E, Q\rangle - k_n \varepsilon_1 \varepsilon_3 \langle E, N\rangle) T.$$

Generally, we can also write;

$$E = \varepsilon_2 \langle E, Q\rangle Q + \varepsilon_3 \langle E, N\rangle N.$$

Then take the derivative of last equation and using Eq(1), we get;

$$\frac{dE}{dt} = (-k_g \varepsilon_1 \varepsilon_2 \langle E, Q\rangle - k_n \varepsilon_1 \varepsilon_3 \langle E, N\rangle) T$$
$$+(\varepsilon_2 \langle E, Q\rangle' - \varepsilon_2 \varepsilon_3 \tau_g \langle E, N\rangle) Q$$
$$+(\varepsilon_3 \langle E, N\rangle' + \varepsilon_2 \varepsilon_3 \tau_g \langle E, Q\rangle) N$$

Finally, we can write the matrix form;

$$\begin{pmatrix} \langle E, Q\rangle' \\ \langle E, N\rangle' \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_3 \tau_g \\ -\varepsilon_2 \tau_g & 0 \end{pmatrix} \begin{pmatrix} \langle E, Q\rangle \\ \langle E, N\rangle \end{pmatrix}.$$

Moreover, since $\langle E,E\rangle = k$, k is a constant and considering the spherical coordinates, we can calculate the following ;

i) If E spacelike and Q timelike, N spacelike, we can get,

$$E = sinh\,\theta\, Q + cosh\,\theta\, N.$$

Then derivating of the last equation, and using Eq (1) and combining the last equation, we can write;

$$\frac{dE}{dt} = cosh\,\theta\, Q + sinh\,\theta\,(-\varepsilon_1 k_g T + \varepsilon_3 \tau_g N) + sinh\,\theta\, N + cosh\,\theta\,(-\varepsilon_1 k_n T - \varepsilon_2 \tau_g Q).$$

We assume that Q timelike and N spacelike;

$$\frac{dE}{dt} = (-\varepsilon_1 k_g \langle E,Q\rangle - \varepsilon_1 k_n \langle E,N\rangle)T + (\frac{d\theta}{dt} + \tau_g)(E \times T)$$

If E spacelike vector and Q spacelike, N timelike vector, we can write

$$E = sinh\,\theta\, N + cosh\,\theta\, Q.$$

Then if we calculate at the same way, we give;

$$\frac{dE}{dt} = (-\varepsilon_1 k_n \langle E,Q\rangle - \varepsilon_1 k_g \langle E,N\rangle)T + (\frac{d\theta}{dt} + \tau_g)\langle E,N\rangle N + (\frac{d\theta}{dt} + \tau_g)\langle E,Q\rangle Q.$$

So, we can write for the two cases;

$$\frac{d\theta}{dt} = -\tau_g.$$

Then, from Fermi-Walker parallelism, we can find that the optical fiber is an E_T – Rytov curve that satisfy $\langle E,T\rangle = 0$. As a result, the direction of the state of polarized light changes in the vector field *T*. Thus, in the optical fiber, the polarization vector is obtained as follows;

$$E = -sinh(\int \tau_g)Q - cosh(\int \tau_g)N.$$

Same way we suppose that E timelike vector, again, we need to examine two situations, these are:
Q timelike vector, N spacelike vector or N timelike, Q spacelike vector. For the first case, we can write;

$$E = cosh\,\theta\, Q + sinh\,\theta\, N.$$

Then derivating of the last equation, and using Eq (1) and combining the last equation, we can write;

$$\frac{dE}{dt} = (-\varepsilon_1 k_g \langle E,N\rangle - \varepsilon_1 k_n \langle E,Q\rangle)T + (\frac{d\theta}{dt} - \varepsilon_2 \tau_g)\langle E,Q\rangle Q + (\frac{d\theta}{dt} + \varepsilon_3 \tau_g)\langle E,N\rangle N.$$

For Q timelike vector, N spacelike vector;

$$\frac{dE}{dt} = (-\varepsilon_1 k_g \langle E,N\rangle - \varepsilon_1 k_n \langle E,Q\rangle)T + (\frac{d\theta}{dt} + \tau_g)(E \times V_1)$$

If E timelike vector and Q spacelike, N timelike vector, we can write

$$E = sinh\,\theta\, Q + cosh\,\theta\, N$$

Then if we calculate at the same way, we give;

$$\frac{dE}{dt} = (-\varepsilon_1 k_n \langle E,N\rangle - \varepsilon_1 k_g \langle E,Q\rangle)T + (\frac{d\theta}{dt} + \tau_g)\langle E,N\rangle Q + (\frac{d\theta}{dt} + \tau_g)\langle E,Q\rangle N.$$

So, we can write for the two cases; in the optical fiber, we must take $\frac{d\theta}{dt} = -\tau_g$. Thus, we can say that the polarization vector E moves the parallel transport along the direction of *T(t)*. With Fermi Walker's parallel transport law, we can express this situation as follows

$$\frac{dE^{FW}}{dt} = \frac{dE}{ds} \pm \langle E,T\rangle \frac{dT}{dt} + \langle E, \frac{dT}{dt}\rangle T$$

Then from Fermi-Walker parallelism, we see that the optical fiber is an E_T – Rytov curve according to $\langle E,T(s)\rangle = 0$. As a result, the direction of the state of polarized light changes in the vector field *T(s)*.

3.2. A geometric phase model of the polarized light wave in the optical fiber through Darboux frame E ⊥ Q

Case 2: Let we suppose that E make a right angle with Q. Thus, we have;

$$\langle E,Q\rangle = 0 \tag{8}$$

If we take the derivative of (8) and take into account the (4) and (1) equations, make the necessary calculations, and assume there is no mechanism lossin the optic fiber because of absorption, we have $\langle E,E\rangle = k$, where k is constant, we can write;

$$E = \varepsilon_1 \langle E,T\rangle T + \varepsilon_3 \langle E,N\rangle N \tag{9}$$

When necessary calculations are made, we can get;

$$\frac{dE}{dt} = \lambda\langle E,N\rangle T + (k_g\varepsilon_1\varepsilon_2\langle E,T\rangle - \tau_g\varepsilon_2\varepsilon_3\langle E,N\rangle)Q$$
$$-\lambda\langle E,T\rangle N \quad (10)$$

The λ part of Eq (12) demonstrates the rotation around the principal vector Q. If we suppose that Q is parallel transported (i.e. $\lambda=0$), then we find;

$$\frac{dE}{dt} = (k_g\varepsilon_1\varepsilon_2\langle E,T\rangle - \tau_g\varepsilon_2\varepsilon_3\langle E,N\rangle)Q.$$

Generally, we can also write;

$$E = \varepsilon_1\langle E,T\rangle T + \varepsilon_3\langle E,N\rangle N.$$

Then take the derivative of last equation and using Eq(1), we get;

$$\frac{dE}{dt} = \left(k_g\varepsilon_1\varepsilon_2\langle E,T\rangle - \tau_g\varepsilon_2\varepsilon_3\langle E,N\rangle\right)Q$$
$$+(\varepsilon_1\langle E,T\rangle' - \varepsilon_1\varepsilon_3 k_n\langle E,N\rangle)T$$
$$+(\varepsilon_3\langle E,N\rangle' + \varepsilon_1\varepsilon_3 k_n\langle E,T\rangle)N$$

Finally, we can write the matrix form;

$$\begin{pmatrix}\langle E,T\rangle' \\ \langle E,N\rangle'\end{pmatrix} = \begin{pmatrix}0 & \varepsilon_3 k_n \\ -\varepsilon_1 k_n & 0\end{pmatrix}\begin{pmatrix}\langle E,T\rangle \\ \langle E,N\rangle\end{pmatrix}.$$

Moreover, since $\langle E,E\rangle = k$, k is a constant and using the spherical coordinates, we can calcuate as following;
If E spacelike and T timelike, N spacelike, we can get;

$$E = \sinh\theta\, T + \cosh\theta\, N.$$

Then derivating of the last equation, and using Eq (1) and combining the last equation, we can write;

$$\frac{dE}{dt} = (\frac{d\theta}{dt} - \varepsilon_1 k_n)\langle E,N\rangle T + (\frac{d\theta}{dt} + \varepsilon_3 k_n)\langle E,T\rangle N$$
$$+(\varepsilon_2 k_g\langle E,T\rangle - \varepsilon_2\tau_g\langle E,N\rangle)Q$$

We assume that T timelike and N spacelike, so we write;

$$\frac{dE}{dt} = (\varepsilon_2 k_g\langle E,T\rangle - \varepsilon_2\tau_g\langle E,N\rangle)Q + (\frac{d\theta}{dt} + k_n)(E\times Q)$$

If E spacelike vector and T spacelike, N timelike vector, we can write

$$E = \sinh\theta\, N + \cosh\theta\, T.$$

Then if we calculate at the same way, we give;

$$\frac{dE}{dt} = (-\varepsilon_2\tau_g\langle E,T\rangle + \varepsilon_2 k_g\langle E,N\rangle)Q + (\frac{d\theta}{dt} + k_n)\langle E,T\rangle T$$
$$+(\frac{d\theta}{dt} + k_n)\langle E,N\rangle N.$$

Same way we suppose that E timelike vector, again, we need to examine two situations, these are: T timelike vector, N spacelike vector or N timelike, T spacelike vector. For the first case, we can write

$$E = \cosh\theta\, T + \sinh\theta\, N$$

Then derivating of the last equation, and using Eq (1) and combining the last equation, we can write; For T timelike vector, N spacelike vector, we get;

$$\frac{dE}{dt} = (\langle E,N\rangle\varepsilon_2 k_g - \varepsilon_2\tau_g\langle E,T\rangle)Q + (\frac{d\theta}{dt} + k_n)(E\times Q).$$

If E timelike vector and T spacelike, N timelike vector, we can write

$$E = \sinh\theta\, T + \cosh\theta\, N$$

Then if we calculate at the same way, we give;

$$\frac{dE}{dt} = (\varepsilon_2 k_g\langle E,T\rangle - \varepsilon_2\tau_g\langle E,N\rangle)Q + (\frac{d\theta}{dt} + k_n)\langle E,N\rangle T$$
$$+(\frac{d\theta}{dt} + k_n)\langle E,T\rangle N.$$

So, we can write for the two cases; in the optical fiber, we must take $\frac{d\theta}{dt} = -k_n$. Thus, we can show that the polarization vector E moves the parallel transport along the direction of Q*(t)*. With Fermi Walker's parallel transport law, we can express this situation as follows

$$\frac{dE^{FW}}{dt} = \frac{dE}{ds} \pm \langle E,Q\rangle\frac{dQ}{dt} + \left\langle E,\frac{dQ}{dt}\right\rangle Q$$

Then, from Fermi--Walker parallelism, we can say that the optical fiber is an $E_Q - Rytov$ curve with the condition that $\langle E,Q\rangle = 0$. As a result, the direction of the state of polarized light changes in the vector field Q. In this way, E is calculated as following;

$$E = -\sinh(\int k_n)T - \cosh(\int k_n)N$$

3.3 A geometric phase model of the polarized light wave in the optical fiber through Darboux frame E ⊥ *N*

Case 3: Let we suppose that *E* make a right angle with *N*. Thus, we have;

$$\langle E,N\rangle = 0 \tag{11}$$

If we take the derivative of (11) and take into account the (4) and (1) equations, make the necessary calculations, and assume there is no mechanism lossin the optic fiber due to absorption, we have $\langle E,E\rangle = k$, where k is constant, we can write;

$$E = \varepsilon_1\langle E,T\rangle T + \varepsilon_2\langle E,Q\rangle Q. \tag{12}$$

When necessary calculations are made, we can get;

$$\frac{dE}{dt} = \lambda\langle E,Q\rangle T - \lambda\langle E,T\rangle Q + (\varepsilon_1\varepsilon_3 k_n\langle E,T\rangle + \tau_g\varepsilon_2\varepsilon_3\langle E,Q\rangle)N \tag{13}$$

The λ part of Eq (15) shows the rotation around the principal normal vector N. If we suppose that N is parallel transported (i.e., λ=0), then we find;

$$\frac{dE}{dt} = (\varepsilon_1\varepsilon_3 k_n\langle E,T\rangle + \tau_g\varepsilon_2\varepsilon_3\langle E,Q\rangle)N.$$

Generally, we can also write;

$$E = \varepsilon_1\langle E,T\rangle T + \varepsilon_2\langle E,Q\rangle Q.$$

Then take the derivative of last equation and considering Eq (1), we get;

$$\begin{aligned}\frac{dE}{dt} &= (\varepsilon_1\langle E,T\rangle' - k_g\varepsilon_1\varepsilon_2\langle E,Q\rangle)T\\ &+(\varepsilon_1\varepsilon_3 k_n\langle E,T\rangle + \tau_g\varepsilon_2\varepsilon_3\langle E,Q\rangle)N\\ &+(\varepsilon_2\langle E,Q\rangle' + \varepsilon_2\varepsilon_1 k_g\langle E,T\rangle)Q.\end{aligned}$$

Finally, we can write the matrix form;

$$\begin{pmatrix}\langle E,T\rangle'\\ \langle E,Q\rangle'\end{pmatrix} = \begin{pmatrix}0 & \varepsilon_2 k_g\\ -\varepsilon_1 k_g & 0\end{pmatrix}\begin{pmatrix}\langle E,T\rangle\\ \langle E,Q\rangle\end{pmatrix},$$

On the other hand, since $\langle E,E\rangle = k$, k is a constant and using the spherical coordinates, we can write the following; If *E* spacelike and *Q* timelike, *T* spacelike, we can write:

$$E = \sinh\theta\, Q + \cosh\theta\, T$$

Then derivating of the last equation, and using Eq(1) and combining the last equation, we can write;

$$\frac{dE}{dt} = (\frac{d\theta}{dt} - \varepsilon_1 k_g)\langle E,Q\rangle T + (\varepsilon_3 k_n\langle E,T\rangle + \varepsilon_3\tau_g\langle E,Q\rangle)N$$
$$+(\frac{d\theta}{dt} + \varepsilon_2 k_g)\langle E,T\rangle Q.$$

We assume that Q timelike and T spacelike;

$$\frac{dE}{dt} = (\varepsilon_3 k_n\langle E,T\rangle + \varepsilon_3\tau_g\langle E,Q\rangle)N + (\frac{d\theta}{dt} - k_g)(E\times N)$$

If E spacelike vector and Q spacelike, T timelike vector, we can write

$$E = \sinh\theta\, T + \cosh\theta\, Q.$$

Then if we calculate at the same way, we give;

$$\frac{dE}{dt} = (\varepsilon_3 k_n\langle E,Q\rangle + \varepsilon_3\tau_g\langle E,T\rangle)N + (\frac{d\theta}{dt} + k_g)\langle E,T\rangle T$$
$$+(\frac{d\theta}{dt} + k_g)\langle E,Q\rangle Q.$$

Same way we suppose that E timelike vector, again, we need to examine two situations, these are:
Q timelike vector, T spacelike vector or T timelike, Q spacelike vector. For the first case, we can write;

$$E = \cosh\theta\, Q + \sinh\theta\, T.$$

Then derivating of the last equation, and using Eq (1) and combining the last equation, we can write;
For Q timelike vector, T spacelike vector;

$$\frac{dE}{dt} = (\varepsilon_3\tau_g\langle E,T\rangle + \varepsilon_3 k_n\langle E,Q\rangle)N + (\frac{d\theta}{dt} - k_g)(E\times N)$$

If E timelike vector and Q spacelike, T timelike vector, we can write

$$E = \sinh\theta\, Q + \cosh\theta\, T$$

Then if we calculate at the same way, we give;

$$\frac{dE}{dt} = (\varepsilon_3\tau_g\langle E,Q\rangle + \varepsilon_3 k_n\langle E,T\rangle)N + (\frac{d\theta}{dt} + k_g)\langle E,Q\rangle T$$
$$+(\frac{d\theta}{dt} + k_g)\langle E,T\rangle Q.$$

So we can write for the two cases; in the optical fiber, we must take $\frac{d\theta}{dt} = -k_g$. Thus, we can say that the polarization vector E moves the parallel transport along the direction of $N(t)$. With Fermi Walker's parallel transport law, we can express this situation as follows

$$\frac{dE^{FW}}{dt} = \frac{dE}{ds} \pm \langle E,N\rangle\frac{dN}{dt} + \langle E,\frac{dN}{dt}\rangle N.$$

Then, from Fermi--Walker parallelism, we see that the optical fiber is an $E_N - Rytov$ curve according to $\langle E, N \rangle = 0$. As a result, the direction of the state of polarized light changes in the vector field N. In this way, the polarization vector is obtained as following

$$E = -sinh(\int k_g)Q - cosh(\int k_g)T$$

4. Example

Let γ is a non-null spacelike curve in Minkowski 3-space defined as;

$$\gamma(t) = (\frac{4}{5}cos((5t), \frac{1}{3}sin(8t) - \frac{4}{3}sin(2t), \frac{1}{3}cos(8t) + \frac{4}{3}cos(2t))$$

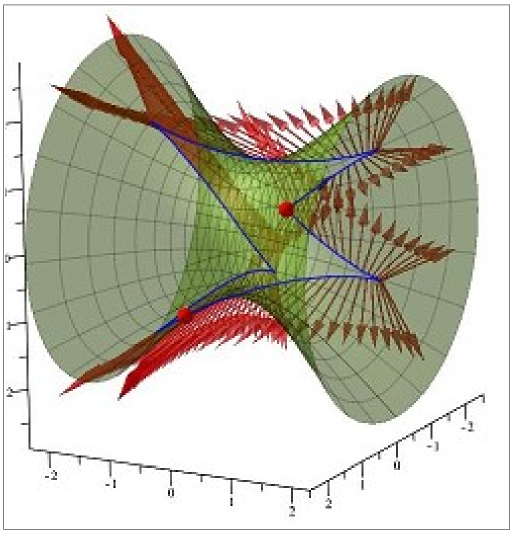

Fig. 1. *Rytov curve (blue) and polarization vector E (red) in the case of E* ⊥ T.

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References

[1] Kravtsov Y.A., and Orlov Y.I., Geometrical Optics of Inhomogeneous Medium, Springer Berlin, Heidelberg (1990) 312.

[2] Rytov S.M., Dokl. Akad. Nauk. SSSR 18, 263 (1938); reprinted in Markovski B., Vinitsky S.I. (eds) Topological Phases in Quantum Theory, World Scientific, Singapore (1989).

[3] Vladimirski V.V., Dokl. Akad. Nauk. SSSR 31, 222 (1941); reprinted in Markovski B., Vinitsky S.I. (eds) Topological Phases in Quantum Theory, World Scientific, Singapore (1989).

[4] Berry M.V., Quantal phase factors accompanying adiabatic changes, Proc. Roy. Soc. London A 392 (1984) 45. https://doi.org/10.1098/rspa.1984.0023

[5] Kugler M., Shtrikman S., Berry's phase, locally inertial frames, and classical analogues, Phys. Rev. D 37:4 (1988) 934. https://doi.org/10.1103/PhysRevD.37.934

[6] Ross J.N., The rotation of the polarization in low briefrigence monomode optical fibres due to geometric effects, Opt. Quantum Electron 16:5 (1984) 455. 1984OptQE..16..455R.

[7] Dandoloff R., Zakrzewski W.J., Parallel transport along a space curve and related phases, J. Phys. A: Math. Gen. 22:11 (1989) L461. https://doi.org/10.1088/0305-4470/22/11/003

[8] Dandoloff R., Berry's phase and Fermi--Walker parallel transport, Phys. Lett. A 139:12 (1989) 19. https://doi.org/10.1016/0375-9601(89)90599-9

[9] Satija I.I., Balakrishnan R., Geometric phases in twisted strips, Phys. Lett. A 373:39 (2009) 3582. https://doi.org/10.1016/j.physleta.2009.07.083

[10] Yamashita O., Effect of the geometrical phase shift on the spin and orbital angular momenta of light traveling in a coiled optical fiber with optical activity, Opt. Commun. 285 (2012) 3740.
https://doi.org/10.1016/j.optcom.2012.05.013

[11] Yamashita O., Geometrical phase shift of the extrinsic orbital angular momentum density of light propagating in a helically wound optical fiber, Opt. Commun. 285 (2012) 3061. https://doi.org/10.1016/j.optcom.2012.02.041

[12] Frins E.M., Dultz W., Rotation of the polarization plane in optical fibers, J. Lightwave Technol. 15:1 (1997) 144. https://doi.org/10.1109/50.552122

[13] Körpinar T., Demirkol R.C., Electromagnetic curves of the linearly polarized light wave along an optical fiber in a 3D semi- Riemannian manifold. Journal of Modern Optics 66:8, (2019) 857-867.
https://doi.org/10.1080/09500340.2019.1579930

[14] Lopez, R., Differential Geometry of Curves and Surfaces in Lorentz-Minkowski Space, Int. Electronic. J. Geom. 7:1 (2014) 44-107.
https://doi.org/10.36890/iejg.594497

[15] Ozdemir, Z., Gok, I., Yaylı Y., Ekmekci, F.N., Notes on Magnetic Curves in 3D semi-Riemannian Manifolds, Turk. J. Math 39 (2015) 412-426.
https://doi.org/10.3906/mat-1408-31

[16] Sunada, T., Magnetic flows on a Riemann surface, In Proceedings of the KAIST Mathematics Workshop: Analysis and Geometry, Taejeon, Korea (1993) 3-6.

[17] Adachi T., Kahler magnetic flow for a manifold of constant holomorphic sectional curvature, Tokyo J. Math 18 (1995) 473. https://doi.org/10.3836/tjm/1270043477

[18] Adachi T., Kahler magnetic on a complex projective space. Proc. Jpn. Acad. Ser. A Math. Sci. 70 (1994) 12. https://doi.org/10.3792/pjaa.70.12

[19] Comtet A., On the Landau Hall levels on the hyperbolic plane, Ann. Phys. 173 (1987) 185. https://doi.org/10.1016/0003-4916(87)90098-4

[20] Cabrerizo J.L., Magnetic fields in 2D and 3D sphere. J. Nonlinear Math. Phys. 20 (2013) 440-450. https://doi.org/10.1080/14029251.2013.855052

[21] Cabrerizo J.L., Fernandez M., Gomez J.S., The contact magnetic flow in 3D Sasakian manifolds, J. Phys. A Math. Theor. 42 (2009) 195201. https://doi.org/10.1088/1751-8113/42/19/195201

[22] Barros M., Magnetic helices and a theorem of Lancret, Proc. Am. Math. Soc. 125:5 (1997) 1503-1509. https://doi.org/10.1090/S0002-9939-97-03692-7

[23] Barros M., Cabrerizo J.L., Fernández M., Romero A., Magnetic vortex flament flows, J. Math. Phys. 48 (2007) 1-27. https://doi.org/10.1063/1.2767535

[24] Druta-Romaniuc S.L., Munteanu M.I., Magnetic curves corresponding to killing magnetic fields in E^3, J. Math. Phys. 52 (2011) 113506. https://doi.org/10.1063/1.3659498

[25] Bozkurt Z., İ. Gök, Y. Yaylı, Ekmekci F.N., A new approach for magnetic curves in 3D Riemannian manifolds, J. Math. Phys. 55 (2014) 053501. https://doi.org/10.1063/1.4870583