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Rytov Curves With Respect To Null Cone Fronts

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In this study, we examined the geometric phase according to Framed curves' frame and researched its relationship to the action of the polarized light wave and electromagnetic trajectories in an optical fiber in Minkowski 3-space. This study consists of four parts. The sections are; The first section is the part where the studies done so far on the subject are given. The second section is the part where the theoretical information used in the publication is included. The third section investigates the geometric phase of the polarization plane of a light wave traveling in an optical fiber through Framed curves' frame in Minkowski 3-space for nullcone fronts of spacelike framed curves. The fourth section shows an example using the Maple program.

Keywords: *Rytov curve, Berry's phase, Magnetic theory, Vector field*

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1. Introduction

In ([1],[2],[3]) basic principles and valuable geometrical descriptions are given which is relation with geometric phase of particle that comes to magnetic field and under this force called Lorentz force. The field, which was initially concerned only with the topological phase, became geometrically interesting with Berry's phase. Ross and Kugler et al. searched the point particle which moves along the optic fiber (see [5],[6]). Other ways, they were researched geometric properties and the result of this phase (see [7],[8],[9],[10],[11]). In ([12],[13]), the authors show the geometric properties of the Berry's phase through the space curve along an optical fiber in a 3dimensional Riemannian manifold, so researchers have defined a space curve as an optic fiber.

Then, in ([15]), researchers studied the motion which is made by the Lorentz force with another perspective. Comtet researched the motion of a charged particle within a constant and static magnetic field in the hyperbolic plane and defined this motion in this space. ([19]). Adachi examined the motion of the charged particle in different space that is complex space ([17],[18]). Cabrerizo et al. defined the new magnetic field in different geometric space ([21]). Along with all the studies done until that time, the most fundamental publications in which magnetic and electromagnetic trajectories were investigated are ([22],[23],[16]). Finally, in ([20],[24],[25]), the authors present different characterizations of the magnetic trajectories.

2. Fundamental backgrounds

Let γ be a spacelike curve. Then, the map

$$(\gamma, V_1, V_2) \ : \ I \to R_3^1 \times \nabla$$

is called a spacelike framed curve if

$$\langle \gamma', V_1 \rangle = 0, \quad \langle \gamma', V_2 \rangle = 0, \quad \forall t \in I,$$

where;

$$\nabla = \{(V_1, V_2) \in S_1^2 \times H_0^2 \mid \langle V_1, V_2 \rangle = 0\}$$

or

$$\nabla = \{(V_1, V_2) \in H_0^2 \times S_1^2 \mid \langle V_1, V_2 \rangle = 0\}.$$

Moreover, the curve γ is called the base curve of the spacelike framed curve. Then $\mu = V_1 \wedge V_2$ is defined, that be a spacelike vector field. On the other hand $\gamma' = a(t)\mu(t)$,

a(t) is a smooth function. It is simple to comprehend that the base curve γ is singular point t_0 if and only if *a(t)=0*. It is denoted;

$$\delta(t) = sign(V_1(t)) = \langle V_1, V_1 \rangle,$$

and the frenet-serret formulas of the frame constructed for such a curve are as indicated in the matrix form below;

$$\begin{pmatrix} V_1' \\ V_2' \\ \mu' \end{pmatrix} = \begin{pmatrix} 0 & l_1 & l_2 \\ l_1 & 0 & l_3 \\ -\delta(t)l_2 & \delta(t)l_3 & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ \mu \end{pmatrix}, \qquad (1)$$

Where

$$\langle V_1', V_2 \rangle = l_1 \quad \langle V_1', \mu \rangle = l_2 \quad \langle V_2', \mu \rangle = l_3.$$

([26]).

So, l_1, I_2 and I_3 are the curvature of the spacelike framed curve.
The magnetic field is a vector field and mathematically corresponds to the vector field div=0 in 3D Riemannian manifolds. The force acting on the magnetic field is called the Lorentz force is defined by the skew symmetric operator ϕ and is given as follows

$$\Phi(X) = V \times X, \qquad (2)$$

The trajectory formed as a result of the particle moving with this force acting on the particle under the influence of the magnetic field is called the magnetic trajectory. The magnetic curves of the magnetic vector field *V* provide the following equation

$$\Phi(t) = V \times t = \nabla_t t, \qquad (3)$$

([22],[23]).

3. A geometric phase model of the polarized light wave in the optical fiber through Null cone fronts

An optical fiber can be defined via a spacelike framed curve in semi-Riemannian manifold. Considering β is a space curve. The direction of the state of the polarized light is defined by the way of the electric field *E*. Thus, along with the optical fiber the direction of *E* can be written as the linear combination of the Framed curve's frame fields in Minkowski 3-space. Then we can write the following

$$\frac{dE}{dt} = \lambda_1 V_1 + \lambda_2 V_2 + \lambda_3 \mu \qquad (4)$$

where λ_i , $i = 1,2,3$ are differentiable functions.
Next, the direction of the state of the polarized light was examined the angle made by the electric field with the fields of the frame in three different cases according to the right angle.

3.1. A Berry phase model of the polarized light wave in the optical fiber through Null cone fronts $\mathbf{E} \perp \boldsymbol{V_1}$

Case 1: In this case we suppose that *E* make a right angle with V_1. So, we can write this expression;

$$\langle E, V_1 \rangle = 0 \qquad (5)$$

If we take the derivative of (5) and take into account the (4) and (1) equations, make the necessary calculations, and supposing there is no mechanism loss in the optical fiber because of absorption, we have $\langle E, E \rangle = k$, k is a constant, we can write;

$$E = \langle E, V_2 \rangle V_2 + \langle E, \mu \rangle \mu \qquad (6)$$

When necessary calculations are made, we can get

$$\frac{dE}{dt} = (-\delta(t)l_1\langle E, V_2 \rangle - \delta(t)l_2\langle E, \mu \rangle)V_1 + (\lambda\langle E, \mu \rangle)V_2 - \lambda\langle E, V_2 \rangle \mu$$

The λ part of Eq(7) shows the rotation around the principal tangent vector V_1 . If we assume that V_1 is parallel transported (i.e. $\lambda = 0$), then we find;

$$\frac{dE}{dt} = (-\delta(t)l_1\langle E, V_2 \rangle - \delta(t)l_2\langle E, \mu \rangle)V_1. \qquad (7)$$

Generally, we can also write;

$$E = \langle E, V_2 \rangle V_2 + \langle E, \mu \rangle \mu.$$

Then take the derivative of last equation and using Eq(1), we get;

$$\frac{dE}{dt} = (l_1\langle E, V_2 \rangle - l_2\delta(t)\langle E, \mu \rangle)V_1 + (\langle E, V_2 \rangle' + \delta(t)l_3\langle E, \mu \rangle)V_2 + (\langle E, \mu \rangle' + l_3\langle E, V_2 \rangle)\mu$$

Finally we can write the matrix form;

$$\begin{pmatrix}\langle E,V_2\rangle' \\ \langle E,\mu\rangle'\end{pmatrix}=\begin{pmatrix}0 & -l_3\delta(t) \\ -l_3 & 0\end{pmatrix}\begin{pmatrix}\langle E,V_2\rangle \\ \langle E,\mu\rangle\end{pmatrix}$$

If we write E with the help of spherical coordinates, we can get as following;
If E spacelike, V 2 timelike,we can get,

$$E = sinh\,\theta\, V_2 + cosh\,\theta\,\mu$$

Then derivating of the last equation, and using Eq(1) and combining the last equation, we can write;

$$\frac{dE}{dt} = cosh\,\theta\, V_2 + sinh\,\theta\,(l_1V_1 + l_3\mu) + sinh\,\theta\,\mu + cosh\,\theta\,(-\delta(t)l_2V_1 + \delta(t)l_3V_2).$$

We assume that V 2 timelike, we get;

$$\frac{dE}{dt} = (l_1\langle E,V_2\rangle - \delta(t)l_2\langle E,\mu\rangle)V_1 + (\frac{d\theta}{dt} + l_3)(E\times V_1).$$

If E timelike vector and V 2 timelike vector, we can write;

$$E = cosh\,\theta\, V_2 + sinh\,\theta\,\mu$$

Then if we calculate at the same way, and using Eq(1) and the last equation, we can write;

$$\frac{dE}{dt} = sinh\,\theta\, V_2 + cosh\,\theta\,(l_1V_1 + l_3\mu) + cosh\,\theta\,\mu + sinh\,\theta\,(-\delta(t)l_2V_1 + \delta(t)l_3V_2).$$

So, we calculate respect to cross product of framed curve's frame, we write;

$$\frac{dE}{dt} = (l_1\langle E,\mu\rangle - \delta(t)l_2\langle E,V_2\rangle)V_1 + (\frac{d\theta}{dt} + l_3)(E\times V_1).$$

So we can write for the two cases; in the optical fiber, we must take $\frac{d\theta}{dt} = -l_3$. Thus, we can say that E moves the parallel transport along the direction of $V_1(t)$. With Fermi Walker's parallel transport law, we can express this situation as follows;

$$\frac{dE^{FW}}{dt} = \frac{dE}{ds} \pm \langle E,V_1\rangle\frac{dV_1}{dt} + \langle E,\frac{dV_1}{dt}\rangle V_1$$

Then from Fermi-Walker parallelism, we see that the optical fiber is an $E_{V_1} -$ Rytov curve with the condition that $\langle E,V_1(t)\rangle = 0$. Consequently, E changes in the vector field $V_1(t)$.

3.2. A Berry phase model of the polarized light wave in the optical fiber through Null cone fronts E ⊥ V_2

Case 2: In this case we suppose that E make a right angle with V_2. So, we can write this expression;

$$\langle E,V_2\rangle = 0 \tag{8}$$

If we take the derivative of (8) and take into account the (4) and (1) equations, we can write;

$$E = \delta(t)\langle E,V_1\rangle V_1 + \langle E,\mu\rangle\mu \tag{9}$$

When necessary calculations are made, we can get

$$\frac{dE}{dt} = \lambda\langle E,\mu\rangle V_1 + (-l_1\langle E,V_1\rangle - l_3\langle E,\mu\rangle)V_2 - \lambda\langle E,V_1\rangle\mu. \tag{10}$$

The λ part of Eq(10) demonstrates the rotation around the principal tangent vector V_2 . If we suppose that V_2 is parallel transported (i.e. $\lambda = 0$), then we find;

$$\frac{dE}{dt} = (-l_1\langle E,V_1\rangle - l_3\langle E,\mu\rangle)V_2.$$

In general, we can write it as;

$$E = \delta(t)\langle E,V_1\rangle V_1 + \langle E,\mu\rangle\mu.$$

Then take the derivative of last equation and using Eq(1), we get;

$$\frac{dE}{dt} = (\delta(t)\langle E,V_1\rangle' - l_2\delta(t)\langle E,\mu\rangle)V_1 + (\delta(t)l_1\langle E,V_1\rangle + \delta(t)l_3\langle E,\mu\rangle)V_2 + (\langle E,\mu\rangle' + \delta(t)l_2\langle E,V_1\rangle)\mu$$

Finally we can write the matrix form;

$$\begin{pmatrix}\langle E,V_1\rangle' \\ \langle E,\mu\rangle'\end{pmatrix}=\begin{pmatrix}0 & l_2 \\ -\delta(t)l_2 & 0\end{pmatrix}\begin{pmatrix}\langle E,V_1\rangle \\ \langle E,\mu\rangle\end{pmatrix}.$$

On the other hand, since $\langle E,E\rangle = k$, k is a constant and using the spherical coordinates, we can write the following;
If E spacelike vector, V_1 timelike vector, we can write;

$$E = sinh\,\theta\, V_1 + cosh\,\theta\,\mu.$$

Then derivating of the last equation, and using Eq(1) and combining the last equation, we can write;

$$\frac{dE}{dt} = cosh\,\theta\, V_1 + sinh\,\theta\,(l_1 V_2 + l_2 \mu) + sinh\,\theta\,\mu + cosh\,\theta\,(-\delta(t) l_2 V_1 + \delta(t) l_3 V_2).$$

We assume that V_1 timelike, we get;

$$\frac{dE}{dt} = (l_1\langle E, V_1\rangle + \delta(t) l_3 \langle E, \mu\rangle) V_2 + (\frac{d\theta}{dt} + l_2)(E \times V_2)$$

If E time like vector and V_1 timelike vector, we can write;

$$E = cosh\,\theta\, V_1 + sinh\,\theta\,\mu$$

Then if we calculate at the same way, and using Eq(1) and the last equation, we can write;

$$\frac{dE}{dt} = sinh\,\theta\, V_1 + cosh\,\theta\,(l_1 V_2 + l_2 \mu) + cosh\,\theta\,\mu + sinh\,\theta\,(-\delta(t) l_2 V_1 + \delta(t) l_3 V_2)$$

So we can write for the two cases; in the optical fiber, we must take $\frac{d\theta}{dt} = l_2$. Thus, we can say that the polarization vector E moves the parallel transport along the direction of $V_2(t)$. Also, this motion can be given through the Fermi-Walker transportation law as follows;

$$\frac{dE^{FW}}{dt} = \frac{dE}{ds} \pm \langle E, V_2\rangle \frac{dV_2}{dt} + \langle E, \frac{dV_2}{dt}\rangle V_2$$

Then from Fermi-Walker parallelism, we see that the optical fiber is an $E_{V_2} -$ Rytov curve with the condition that $\langle E, V_2(t)\rangle = 0$. Consequently, the direction of the state of polarized light changes in the vector field $V_2(t)$.

3.3 A Berry phase model of the polarized light wave in the optical fiber through Null cone fronts E $\perp$ μ

Case 3: In this case we suppose that E make a right angle with μ. So, we can write this expression;

$$\langle E, \mu\rangle = 0 \qquad (11)$$

If we take the derivative of (8) and take into account the (4) and (1) equations, we can write;

$$E = \delta(t)\langle E, V_1\rangle V_1 + \langle E, V_2\rangle V_2 \qquad (12)$$

When necessary calculations are made, we can get

$$\frac{dE}{dt} = (\lambda\langle E, V_2\rangle) V_1 - \lambda\langle E, V_1\rangle V_2 + (\delta(t) l_2 \langle E, V_1\rangle - \delta(t) l_3 \langle E, V_2\rangle)\mu \qquad (13)$$

The λ part of Eq(13) demonstrates the rotation around the principal tangent vector μ . If we suppose that π is parallel transported (i.e. $\lambda = 0$), then we find;

$$\frac{dE}{dt} = (\delta(t) l_2 \langle E, V_1\rangle - \delta(t) l_3 \langle E, V_2\rangle)\mu$$

Generally, we can also write;

$$E = \delta(t)\langle E, V_1\rangle V_1 + \langle E, V_2\rangle V_2$$

Then take the derivative of last equation and using Eq(1), we get;

$$\frac{dE}{dt} = (\delta(t)\langle E, V_1\rangle' + l_1\langle E, V_2\rangle) V_1 + (\delta(t) l_1 \langle E, V_1\rangle + \langle E, V_2\rangle') V_2 + (\delta(t) l_2 \langle E, V_1\rangle + l_3 \langle E, V_2\rangle)\mu$$

Finally, we can write the matrix form;

$$\begin{pmatrix} \langle E, V_1\rangle' \\ \langle E, V_2\rangle' \end{pmatrix} = \begin{pmatrix} 0 & -l_1\delta(t) \\ -l_1\delta(t) & 0 \end{pmatrix} \begin{pmatrix} \langle E, V_1\rangle \\ \langle E, V_2\rangle \end{pmatrix}$$

On the other hand, since $\langle E, E\rangle = k$, k is a constant and using the spherical coordinates, we can write the following; If E , V 2 are respectively spacelike and timelike vector, we get;

$$E = sinh\,\theta\, V_2 + cosh\,\theta\, V_1$$

We assume that V 2 timelike, we get;

$$\frac{dE}{dt} = (l_3\langle E, V_2\rangle + l_2\langle E, V_1\rangle)\mu + (\frac{d\theta}{dt} + l_1)(E \times \mu)$$

If E timelike vector and V 2 timelike vector, we can write;

$$E = cosh\,\theta\, V_2 + sinh\,\theta\, V_1$$

Then if we calculate at the same way, and use Eq(1) and the last equation, we can write;

$$\frac{dE}{dt} = sinh\,\theta\, V_2 + cosh\,\theta\,(l_1 V_1 + l_3 \mu) + cosh\,\theta\, V_1 + sinh\,\theta\,(l_1 V_2 + l_2 \mu)$$

So, we calculate respect to cross product of framed curve's frame, we write;

$$\frac{dE}{dt} = (l_2\langle E, V_2\rangle + l_3\langle E, V_1\rangle)\mu + (\frac{d\theta}{dt} + l_1)(E \times \mu)$$

So, we can write for the two cases; in the optical fiber, we must take $\frac{d\theta}{dt} = l_1$. Thus, we can say that the polarization vector E moves the parallel transport along the direction of $\mu(t)$. Also, this motion can be given through the Fermi-Walker transportation law as follows;

$$\frac{dE^{FW}}{dt} = \frac{dE}{ds} \pm \langle E, \mu \rangle \frac{d\mu}{dt} + \langle E, \frac{d\mu}{dt} \rangle \mu$$

Then from Fermi-Walker parallelism, we see that the optical fiber is an $E_\mu -$ Rytov curve with the condition that $\langle E, \mu(t) \rangle = 0$. Consequently, the direction of the state of polarized light changes in the vector field $\mu(t)$.

4. Example

Let γ be a spacelike curve in R_3^1, defined by;

$$\gamma(t) = (\frac{1}{2}t^2, \frac{1}{2}t^2, \frac{1}{4}t^4 + \frac{1}{2}t^2)$$

$$V_1(t) = \frac{1}{\sqrt{(t^2+1)^2((t^2+1)^2+1)}}(0, t^2+1, -1)$$

$$V_2(t) = \frac{1}{\sqrt{(t^2+1)^2((t^2+1)^2+1)}}((t^2+1)^2+1, 1, t^2+1)$$

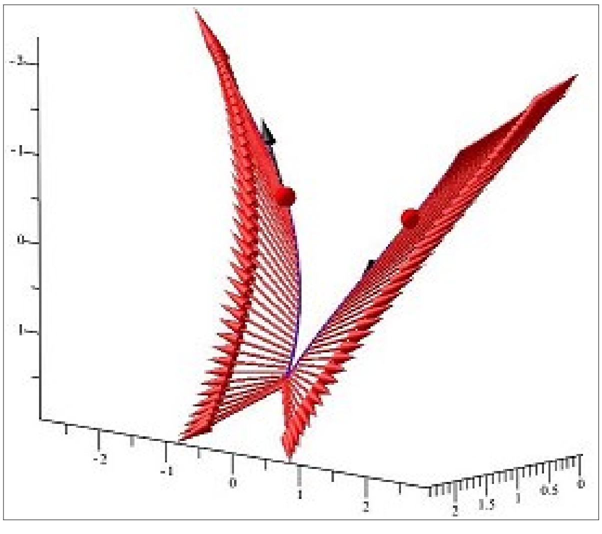

Fig. 1. Behavior of the polarization E (red) along the optical fiber for the case E ⊥ V_1

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